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<td>Author(s)</td>
<td>Hsu, Jui-Pang; 許，瑞邦; Takaharu, Hiraoka; 平岡，隆晴; Manabu, Inoue; 井上，學</td>
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Analysis of Eigenmode for 3-D Optical Waveguide based on Planar Circuit Equations and Lateral Equivalent Network

Manabu Inoue  Takaharu Hiraoka  Hsu, Jui-Pang
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Abstract—Exact calculation of eigenvalues and eigenmodes for 3-D optical waveguide are important for rigorous analysis/design of optical waveguide circuit. Based on planar circuit equations and mode expansion method, equivalent network for 3-D optical waveguide in lateral direction is derived and applied to the analysis of practical 3-D optical waveguide with success.

Index Terms—3-D optical waveguide, planar circuit equations, lateral equivalent network, eigenmode

1. Introduction
3-D optical waveguide as shown in Fig.1 is key elements for integration of optical circuit. Hence, exact calculation of eigenmode for these waveguide is important for analysis/design of optical circuit. In this paper, rigorous equivalent network in lateral direction for 3-D optical waveguide is derived, based on surface-wave planar circuit equations and modal analysis, and applied to the analysis of raised strip line structure shown in Fig.1(a) with success.

2. Planar Circuit Equations for Surface-Wave
Surface-wave planar circuit shown in Fig.2 was proposed and the corresponding surface-wave planar circuit equations have been formulated[1]. Moreover, in order to make the continuous spectrum to discrete mode, we place metal roof and floor as shown in Fig.2(b). The fields of TE/TM mode in the planar circuit are given by product of x-y dependent function and z dependent function as shown in Table 1 by applying separation of variables technique, where relative dielectric constant \( \epsilon_z \) is given as a function of height z. The z dependent functions are eigenmode functions in height direction (height mode). Then, planar electric or magnetic voltage and current density in Fig.2(a) defined in the top of Table 2 are related by planar circuit equations for TE/TM mode given in the middle of Table 2, where planar series/shunt immittance and the corresponding planar characteristic admittance and propagation constant are defined in the bottom of Table 2. \( \beta \) in Table 1 is eigenvalue for corresponding planar propagation constant and given by solving z dependent function under boundary conditions.

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**Table 1** Separation of variables for field distribution of surface-wave planar circuit \( \eta = \sqrt{\mu / \epsilon_z} \), \( \omega = \omega_0 / \epsilon_z \)

<table>
<thead>
<tr>
<th>Separation of variables</th>
<th>x-y dependent function</th>
<th>z dependent function</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>TE mode</strong></td>
<td>( \mathbf{E}^H(x,y,z) = \mathbf{E}^H(x,y) f^H_m(z) )</td>
<td>( \mathbf{H}^E(x,y,z) = \mathbf{H}^E(x,y) k^E_m(z) )</td>
</tr>
<tr>
<td>( \mathbf{E}^H(x,y,z) = \mathbf{E}^H(x,y) f^H_m(z) )</td>
<td>( \nabla \times (\mathbf{k} \times \mathbf{E}^H(x,y)) = -j \omega \mu f^H_m(z) \mathbf{H}^E(x,y) )</td>
<td>( \frac{\partial \mathbf{E}^H_m(z)}{\partial z} = \gamma^H_m \mathbf{E}^H_m(z) )</td>
</tr>
<tr>
<td><strong>TM mode</strong></td>
<td>( \mathbf{H}^E(x,y,z) = \mathbf{H}^E(x,y) k^E_m(z) )</td>
<td>( \mathbf{E}^H(x,y,z) = \mathbf{E}^H(x,y) f^H_m(z) )</td>
</tr>
<tr>
<td>( \mathbf{H}^E(x,y,z) = \mathbf{H}^E(x,y) k^E_m(z) )</td>
<td>( \nabla \times (\mathbf{k} \times \mathbf{H}^E(x,y)) = -j \omega \epsilon f^E_m(z) \mathbf{E}^H(x,y) )</td>
<td>( \frac{\partial \mathbf{H}^E_m(z)}{\partial z} = \gamma^E_m \mathbf{H}^E_m(z) )</td>
</tr>
</tbody>
</table>

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**Table 2** Planar circuit equations for TE/TM surface-waves

<table>
<thead>
<tr>
<th>TE mode</th>
<th>TM mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{V}^H(x,y) = -d \cdot \mathbf{H}^E(x,y) [m] )</td>
<td>( \mathbf{V}^E(x,y) = -d \cdot \mathbf{E}^H(x,y) [m] )</td>
</tr>
<tr>
<td>( \mathbf{J}^H(x,y) = \mathbf{k} \times \mathbf{E}^H(x,y) [V/m] )</td>
<td>( \mathbf{J}^E(x,y) = \mathbf{H}^E(x,y) \times \mathbf{k} [A/m] )</td>
</tr>
</tbody>
</table>

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**Fig.1 Examples of 3-D optical waveguide structure**

**Fig.2 Surface-wave planar circuit**

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**Fig.3 Examples of 3-D optical waveguide surface-waves**

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**Fig.4 Examples of 3-D optical waveguide surface-waves**
condition in height direction. Once planar circuit equations are solved under given excitation and boundary condition, the fields in planar circuit are given by summation of TE/TM modes given in Table 3. In order to analyze optical planar circuit based on microwave circuit concept, exact calculation of eigenvalue (propagation constant or effective refractive index) and eigenmode of 3-D optical waveguide becomes important. Calculation is carried out by lateral equivalent network, which is derived in the following section.

3. Derivation of Equivalent Network for 3-D waveguide

Cross section of 3-D optical waveguide shown in Fig. 1 consists of three uniform regions and two step discontinuities in lateral direction as shown in Fig. 3(a). Also electrical wall is prepared at both sides. Here, half structure of optical waveguide as shown in Fig. 3(b) is analyzed because of symmetrical structure.

(1) Equivalent network for uniform region

Assuming that the field varies as $e^{-j \omega t}$ along waveguide, $x$ dependent planar voltage/current density of each mode is given by following transmission line equations (1), where $V(x,y)=V(x)e^{-j\beta_y y}$ and $J(x,y)=(J_x(x),J_y(x))e^{-j\beta_y y}$ are assumed.

$$\begin{align}
\frac{dV(x)}{dx} &= -j\beta_y J(x) \\
\frac{dJ(x)}{dx} &= -j\beta_y^2 V(x) \quad \beta_y^2 = \beta_x^2 - \beta_y^2
\end{align}$$

(1)

Hence, the equivalent network for uniform region is given by multi-transmission line of TE/TM as shown in Fig. 3(c), whose network parameters (phase constant $\beta$ and characteristic admittance $y_c$, in $x$-direction) are defined in Table 4. When characteristic admittance along waveguide $y_c$ is defined as in Table 4, current density of any mode along waveguide is given by eq.(2).

$$J_{p,\beta}^m(x) = J_{p,\beta}^m(x) = V_{p,\beta}^m (x) , J_{p,\beta}^m(x) = J_{p,\beta}^m(x) = V_{p,\beta}^m (x)$$

(2)

(2) Equivalent network for step discontinuity

Continuity of tangential field at step discontinuity gives following mode coupling equations (3),(4) at the step.

$$\begin{align}
\sum_{n} \frac{J_{p,\beta}^m(x) f_{m,n}^p (z) + j \eta e \sum_{n} J_{p,\beta}^m(x) h_{m,n}^p (z)}{\beta_{\beta,m}^p} - \sum_{n} J_{p,\beta}^m(x) f_{m,n}^p (z) &= \sum_{n} \sum_{q} J_{p,\lambda}^m(x) f_{m,q}^p (z) \\
\frac{j \eta e \sum_{n} J_{p,\beta}^m(x) h_{m,n}^p (z) + \sum_{n} J_{p,\beta}^m(x) f_{m,n}^p (z)}{\beta_{\beta,m}^p} - \sum_{n} J_{p,\beta}^m(x) h_{m,n}^p (z) &= \sum_{n} \sum_{q} J_{p,\lambda}^m(x) h_{m,q}^p (z)
\end{align}$$

(3)

(4)

Where mode coupling coefficients are defined in table 5. When parallel current given by eq.(2) is substituted into eq.(4), then current relations (4) are replaced by eq.(5).

Table 3 Field description in i-th region by mode summation

<table>
<thead>
<tr>
<th>$E_{i}^m$</th>
<th>$H_{i}^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_n J_{m,n}^p(x,y) f_{m,n}^p (z) + j \eta e \sum_n J_{m,n}^p(x,y) h_{m,n}^p (z)$</td>
<td>$-\frac{1}{j \eta e} \sum_n J_{m,n}^p(x,y) f_{m,n}^p (z) - \sum_n J_{m,n}^p(x,y) f_{m,n}^p (z)$</td>
</tr>
<tr>
<td>$\sum_n J_{m,n}^p(x,y) f_{m,n}^p (z) + j \eta e \sum_n J_{m,n}^p(x,y) h_{m,n}^p (z)$</td>
<td>$-\frac{1}{j \eta e} \sum_n J_{m,n}^p(x,y) h_{m,n}^p (z) + \sum_n J_{m,n}^p(x,y) f_{m,n}^p (z)$</td>
</tr>
</tbody>
</table>

Table 4 Network parameters for multimode transmission line

<table>
<thead>
<tr>
<th>Mode</th>
<th>TE</th>
<th>TM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{\beta,m}^T$</td>
<td>$\beta_{\beta,m}^T$</td>
<td>$\beta_{\beta,m}^T$</td>
</tr>
<tr>
<td>$\beta_{\beta,m}^T$</td>
<td>$\beta_{\beta,m}^T$</td>
<td>$\beta_{\beta,m}^T$</td>
</tr>
<tr>
<td>$k_{\eta} = \omega \sqrt{e_\epsilon} \mu$</td>
<td>$\eta e \mu / \epsilon_\epsilon$</td>
<td>$\mu / \epsilon_\epsilon$</td>
</tr>
<tr>
<td>$\gamma_{C,m} = \eta e \mu / \epsilon_\epsilon$</td>
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</tr>
</tbody>
</table>

Table 5 Mode couple coefficient at step discontinuity

<table>
<thead>
<tr>
<th>Neighboring</th>
<th>Inside</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{m,n}^T(E(z), z f_{m,n}^T (z)) dz$</td>
<td>$F_{m,n}^T(E(z), z f_{m,n}^T (z)) dz$</td>
</tr>
<tr>
<td>$F_{m,n}^T(E(z), z f_{m,n}^T (z)) dz$</td>
<td>$H_{m,n}^T(E(z), z f_{m,n}^T (z)) dz$</td>
</tr>
<tr>
<td>$H_{m,n}^T(E(z), z f_{m,n}^T (z)) dz$</td>
<td>$H_{m,n}^T(E(z), z f_{m,n}^T (z)) dz$</td>
</tr>
</tbody>
</table>

Fig. 3 3-D optical waveguide structure, equivalent network in lateral direction for half structure and vector representation.
Each mode converted current \( i \) in eq.(5) is given by eq.(6), where \( i \) is region number and \( k \) is port number.

\[
\hat{J}^{(1,2)}_m = \sum_{l} \gamma^{H(E)}_{l,n} \hat{J}^{(1,2)}_{m,l} \quad \hat{J}^{(1,2)}_m = \sum_{l} \gamma^{H(E)}_{l,n} \hat{J}^{(1,2)}_{m,l} \quad (i,k = 1,2) \tag{6}
\]

(6) TE-TM mode conversion admittance given by

\[
\gamma^{H(E)}_{l,n} = j \hat{V}^{H(E)}_{l,n} / j \hat{I}^{H(E)}_{l,n} \quad (i,j = 1,2) \tag{7}
\]

Eq.(3) and (5) give equivalent network for step discontinuity, which is given by multiport ideal transformer and mode converted current source as shown in Fig.3(c). Therefore, whole equivalent network for half structure in lateral direction is given by Fig.3(c) or Fig.3(d) in vector representation. Matrix notation corresponding vector representation is now used to simplify the following analysis.

4. Formulation of Eigenvalue Equation for 3-D Waveguide

Mode matching technique is introduced for formulation of eigenmode equation. Since mode admittance looking toward left side (outside) at port (1,2) is easily derived from equivalent network in Fig. 3 and given by eq.(7) and (8).

\[
\begin{bmatrix}
\hat{J}^{(1,2)}_1 \\
\hat{J}^{(1,2)}_2
\end{bmatrix} =
\begin{bmatrix}
\hat{Y}^{1,2}_{1,0} & 0 \\
0 & \hat{Y}^{1,2}_{2,0}
\end{bmatrix}
\begin{bmatrix}
\hat{V}^{1,2}_1 \\
\hat{V}^{1,2}_2
\end{bmatrix} \quad \hat{J}^{(1,2)}_1 = \hat{Y}^{1,2}_1 \hat{V}^{1,2}_1 \tag{8}
\]

where input mode admittance \( \hat{Y}^{1,2}_{1,0}, \hat{Y}^{1,2}_{1,0}, \hat{Y}^{1,2}_{2,0}, \hat{Y}^{1,2}_{2,0} \) are defined in Table 6. Hence the mode matching equation becomes

\[
\begin{bmatrix}
\hat{J}^{(1,2)}_1 \\
\hat{J}^{(1,2)}_2
\end{bmatrix} = 0 \quad \left( \hat{Y}_{1,0} + \hat{Y}_{2,0} \right) \hat{V}^{(2,1)} = 0 \tag{11}
\]

which is eigenvalue equation. The eigenvalue \( \beta_l \) gives effective refractive index of each mode and eigenvector gives field distribution of the corresponding mode by using the result of Table 3 and \( \varepsilon_l \) dependency.

5. Calculated Results

Eigenvalues (propagation constant, i.e. effective refractive index \( n_{eff} \)) and eigenmodes (field distribution of normal mode) of raised strip line 3-D optical waveguide shown in Fig.3 are calculated based on the lateral equivalent network. The dimension and parameters of the structure analyzed are given as \( n = 1.49, \quad n = 1.47, \quad d_L = 1.0[\mu m], d_T = 2.0[\mu m], \quad d_p = 4.0[\mu m], \quad W^{(01)} = W/2[\mu m], \quad W^{(02)} = 2.0[\mu m]-W^{(01)}[\mu m]. \)

Table 6 Input mode admittance matrix at port (1,2) and port (2,1)

<table>
<thead>
<tr>
<th>Electric wall</th>
<th>Magnetic wall</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{Y}^{1,2}_{1,0} )</td>
<td>( \hat{Y}^{1,2}_{2,0} )</td>
</tr>
<tr>
<td>( \hat{Y}^{2,2}_{1,0} )</td>
<td>( \hat{Y}^{2,2}_{2,0} )</td>
</tr>
<tr>
<td>( \hat{Y}^{3,2}_{1,0} )</td>
<td>( \hat{Y}^{3,2}_{2,0} )</td>
</tr>
<tr>
<td>( \hat{Y}^{4,2}_{1,0} )</td>
<td>( \hat{Y}^{4,2}_{2,0} )</td>
</tr>
</tbody>
</table>

Fig.4 Calculated effective refractive index \( n_{eff} \) (solid line:guided mode, dotted line:substrate mode, broken line:air mode)
(1) Convergence of \( n_{eff} \) with number of height mode
Convergence behavior of eigenvalue, i.e. \( n_{eff} \), for the dominant mode with number of TE/TM height mode is calculated and shown in Fig.6, which demonstrates that \( n_{eff} \) is surely converged at around 20 TE/TM height modes. 3-D optical waveguide dimension is shown in Fig.6. Through our calculation, 50 TE/TM height modes are taken into consideration.

(2) Calculation of effective refractive index (\( n_{eff} \))
Calculated effective refractive index \( n_{eff} \) vs. width of waveguide \( W \) are shown in Fig.4, where modes can be classified as guided mode, substrate mode, and air mode depending on value of effective refractive index.

(3) Field distribution of each eigenmode
The field distribution of any eigenmode can be easily calculated by obtaining eigenvector of eq.(11). Through our calculation, field strength is \([V/\mu m]\) for 1[Watt] input and magnetic field is multiplied by \( \eta_o = \sqrt{\mu_o / \varepsilon_0} [\Omega] \). As examples of calculated eigenmode field distribution, that of guided dominant mode are shown in Fig.5 and typical substrate mode and air mode are shown Fig.7. Each mode is corresponding to points [a],[b] and [c] in Fig.4. The maximum values of each field component are shown in each figure for 1[Watt] input.

7. Conclusion
We formulated rigorous lateral equivalent network for 3-D optical waveguide based on surface wave planar circuit equations and mode analysis. The derived equivalent network is practically applied to 3-D optical waveguide with success.

Fig.6 Convergence of \( n_{eff} \) with number of TE/TM mode
Advantages of this method are (1) Physical situation in waveguide becomes clear by mode analysis and (2) main calculation error is caused by truncation of mode, which will give the reasonable estimation of error vs. CPU time relation. Thus derived equivalent network will be useful for analysis of any other 3-D optical waveguide.

Reference